Journal of Mathematical Behavior 81 (2026) 101300

Contents lists available at ScienceDirect

Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Realistic visual representations as mediators between everyday and mathematical discourses in heterogeneous classrooms

Merav Weingarden [a,b,*], Ronnie Karsenty [b], Boris Koichu [b]

[a] Tel Aviv University, School of Education, Haim Lebanon St. 55, Tel Aviv 6997801, Israel
[b] Weizmann Institute of Science, The Department of Science Teaching, Herzl St. 234, Rehovot 7610001, Israel

ARTICLE INFO

Keywords:
Realistic representations
Mathematics teaching
Heterogeneous classrooms
Professional development
Commognition

ABSTRACT

Creating effective learning environments in heterogeneous mathematics classrooms remains a persistent challenge. This exploratory study examines how realistic representations can support learners across achievement levels. Through a two-part investigation, the study first conducts an a priori analysis of two representation-oriented activities to articulate the roles of realistic representations and then explores how teachers view such roles as supporting low- and high-achieving students. The initial analysis identifies four roles of realistic representations: *motivating*, *initiating*, *connecting*, and *applying*—each mediating differently between everyday and mathematical discourses. The subsequent examination through case studies of two teachers implementing these activities in their classrooms shows that one teacher recognized motivational benefits for low-achieving students, whereas the other highlighted their value in encouraging connection-making for high-achieving students. We discuss how explicating the multifaceted roles of realistic representations can help teachers and teacher educators design more inclusive mathematics instruction.

1. Introduction

Grouping mathematics students by ability versus teaching mathematics in mixed-ability settings (often termed tracking vs. detracking) has been a long-standing controversy in educational research (Gamoran, 2010; Slavin, 1990). While some studies suggest that tracking helps teachers tailor instruction to students' needs (e.g., Mulkey et al., 2005) and can yield faster progress and higher achievement for students in higher-level tracks (Jong et al., 2004; Schofield, 2010), others criticize tracking for reasons associated with equity and stratification concerns. These issues include lower-quality instruction and fewer learning opportunities for students in lower tracks, limited mobility between tracks, stigmatization, and reduced diversity of perspectives in classrooms (Boaler, 2008; Burris & Garrity, 2008; Oakes, 2005; Weingarden & Heyd-Metzuyanim, 2021, 2023; Zevenbergen, 2003).

Despite documented cognitive, affective, and social benefits of heterogeneous learning environments (e.g., Boaler et al., 2000), teaching in such settings—and personalizing instruction to students' needs, prior knowledge, and skills—remains challenging for teachers (Gamoran, 2004; Gibbs & McKay, 2021). Accordingly, there is a critical need for effective strategies and resources to support teachers in this endeavor (Bardy et al., 2021; Lenz et al., 2024).

One promising approach to address these challenges is the deliberate use of diverse representations in mathematics instruction (Dreyfus, 1991; Goldin, 2020; Goldin & Shteingold, 2001; Taylor & Dyer, 2014). Representations include mathematical language,

* Corresponding author at: Tel Aviv University, School of Education, Haim Lebanon St. 55, Tel Aviv 6997801, Israel.
E-mail address: weingardenm@tauex.tau.ac.il (M. Weingarden).

https://doi.org/10.1016/j.jmathb.2025.101300
Received 28 January 2025; Received in revised form 2 November 2025; Accepted 10 November 2025
Available online 22 November 2025
0732-3123/© 2025 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY-NC license (http://creativecommons.org/licenses/by-nc/4.0/).

symbolic notations (e.g., equations, formulas), verbal descriptions, graphic and visual displays (e.g., diagrams, charts, graphs), and numerical expressions, as well as realistic representations such as concrete models or manipulatives, contextual situations, and pictorial illustrations (Kumar et al., 2017; Lesh et al., 1987; Otten et al., 2019; Tripathi, 2008; Van den Heuvel-Panhuizen, 2003). Research has consistently shown that using diverse representations can deepen students' understanding of mathematical ideas, especially around abstract concepts (Ainsworth, 2006; Duval, 2006; Warren & Cooper, 2009). Diverse representations can offer alternative ways of interpreting, communicating, and reasoning about abstract mathematical ideas and provide students with more accessible ways to think about and understand mathematical concepts. In the context of heterogeneous classrooms, using various representations is particularly crucial since different students find different representations more accessible than others to communicate about mathematical ideas and develop a deeper understanding around them (e.g., Karsenty et al., 2007).

Using realistic representations—for example a balance scale to communicate the idea of an equation and its solution process, or a mall elevator to communicate signed numbers (Fig. 1)—has been found particularly beneficial for low-achieving students, who often engage more effectively through concrete, contextualized pathways into abstract concepts (Fyfe et al., 2015; Otten et al., 2019; Stein & Bovalino, 2001; Tripathi, 2008; Van den Heuvel-Panhuizen, 2003; Witzel et al., 2003).

However, little is known about how these representations can be used and implemented in heterogeneous classroom contexts and how teachers perceive their roles in addressing diverse learners' needs. This exploratory study addresses that gap by investigating the roles of realistic representations in activities intended for heterogeneous classrooms and by examining how teachers view these roles when implementing such activities.

To examine realistic representations and their role in learning mathematics—specifically in heterogeneous classrooms—we adopt the commognitive framework (Sfard, 2008), viewing realistic representations as communicational tools that can bridge everyday and mathematical discourses. While multiple theoretical perspectives inform work on representations, the commognitive approach offers advantages for this investigation because it treats representations as external, observable communicational tools within mathematical discourse, enabling analysis across contexts (from written instructional materials to spoken reflections). In the following sections, we review existing approaches to mathematical representations and situate our theoretical choice; we then review teachers' practices with representations in heterogeneous classrooms and finally elaborate how realistic representations can serve as mediators between everyday and mathematical discourses.

2. Literature review

2.1. *Mathematical representations as mental constructs and as tools for communication*

Mathematical representations, as defined by Goldin (1998, 2020), are:

> Visible or tangible productions – such as diagrams, number lines, graphs, arrangements of concrete objects or manipulatives, physical models, written words, mathematical expressions, formulas and equations, or depictions on the screen of a computer or calculator – that encode, stand for, or embody mathematical ideas or relationships (Goldin, 2020, p. 566).

Goldin distinguishes between *internal* representations—"a person's mental, cognitive, or brain constructs, concepts, or configurations"—and *external* representations, which are "external to the individual who produced them and accessible to others for observation, discussion, interpretation, and/or manipulation." (p. 566).

Dreyfus (1991), similarly to Goldin, views representations as mental constructs that individuals create when thinking about mathematical objects or processes. Different people may have different mental representations of the same concept, leading to discrepancies in understanding, especially between students and teachers. Dreyfus emphasizes the importance of having multiple

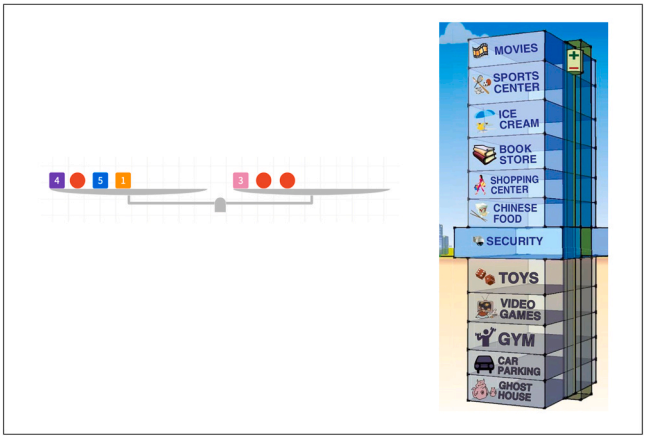


Fig. 1. Examples of visual realistic representations. Left-hand side: a scale representing an equation (Robotti et al., 2023). Right-hand side: a mall elevator representing a number line (Kumar et al., 2017).

representations for a single mathematical concept, allowing for richer understanding and enhancing the flexible translation between different forms (e.g., graphical, algebraic) as needed for advancing mathematical thinking. Duval (2006, 2020), drawing on semiotics (the study of signs and symbols), introduces "registers of semiotic representation," i.e., systems of signs (graphical, symbolic, verbal) that are used to convey the meaning of mathematical ideas. Each register has its own rules and constraints.

Despite the acknowledged importance of multiple representations and their affordances for student learning, a critical element that often goes beyond the representations themselves is the flexibility with which students can transform, substitute, and switch between various forms of representations, such as symbolic, graphical, verbal, and numerical (Ainsworth, 2006; Kilpatrick et al., 2001; Lesh et al., 1987; Tripathi, 2008). This element, known as "representations as process" (NCTM, 2000), "translation of various representations" (Goldin & Shteingold, 2001), or "saming realizations" (Sfard, 2008; Weingarden & Heyd-Metzuyanim, 2023), is pivotal for developing a robust understanding of mathematical concepts. Dreyfus (1991) cautioned that simply having many representations is insufficient; what matters is that they are "correctly and strongly linked" (p. 32). Duval (2006) likewise stresses that "what matters is not representations but their transformation" (p. 107), which consists of "substituting one semiotic representation B for another A, if both representations denote the same object" (Duval, 2020, p. 724).

While these transformations and translations between representations are important for mathematical understanding, when examining how they occur in classroom interactions—between students, teachers, and mathematical tasks—it becomes useful to view representations not only as internal mental constructs but also as external communicational tools. In interactive classroom contexts, transformations between representations can be observed as communicative acts where representations serve as shared objects that participants can discuss, manipulate, and transform together. This perspective aligns with research that frames representations as modes of communicating mathematical ideas in social activity. Representing can thus be viewed as a practice that is socially mediated (Kaput et al., 2008; Selling, 2016; Roth & McGinn, 1998), where representations are "tools for communicating about mathematics" (Fonger, 2019, p. 2). As stated by the NCTM, students at all levels need to "create and use representations to organize, record, and communicate mathematical ideas" (NCTM, 2000, p. 64).

Two productive ways to enhance mathematical communication are contextualizing mathematics in students' everyday lives (e.g., shopping scenarios for percentages and discounts) and concretizing abstract concepts in tangible models (e.g., using a balance scale to teach equations). Within the communicational perspective described above, both contextualization and concretization can be achieved through realistic representations—those derived from situations that are experientially real to students (Boaler, 1993; Gravemeijer, 1999; Gravemeijer & Doorman, 1999; Van den Heuvel-Panhuizen, 2003). Such representations provide accessible entry points to mathematical discourse by grounding abstract concepts in students' lived experiences.

2.2. Using realistic representations with students of different achievement levels: affordances, challenges, practices, and teacher perspectives

In heterogeneous classrooms, teachers often face challenges related to differentiation, engagement across ability levels, and equitable access to meaningful learning (Boaler, 2008; Boaler & Sengupta-Irving, 2016; Eikeland & Ohna, 2022; Gardesten & Palmér, 2023; Goyibova et al., 2025; Prast et al., 2018; Rubin & Noguera, 2004; Tomlinson, 2001). Using realistic representations is particularly important in addressing these challenges (Dreher & Kuntze, 2015; Flores, 2010; Taylor & Dyer, 2014). It can support low-achieving students who often struggle to develop conceptual understanding of abstract mathematics and need alternative, more concrete, context-based, and visual representations (Baccaglini-Frank, 2021; Hodgen et al., 2024; Karsenty et al., 2007; Xin et al., 2022), as well as high-achieving students, for whom translating among representations is itself a productive, cognitively engaging activity (Dreher & Kuntze, 2015; Lepak et al., 2018; Stylianou, 2010).

Low-achieving students often face significant challenges in learning mathematics. These students may struggle with abstract thinking, working memory, symbol manipulation, and problem solving (Swanson & Jerman, 2006; Tall & Razali, 1993; Xin et al., 2022). Studies in mathematics and special education suggest that concrete models, manipulatives, and visual images from everyday life provide a tangible, contextualized foundation for mathematical concepts (Flores, 2010; Kilpatrick et al., 2001; Otten et al., 2019; Subramaniam, 2019; Witzel et al., 2003). Such studies emphasize the importance of using concrete representations in the early stages of concept development. The Concrete-Representational-Abstract (CRA) sequence (Flores, 2010), for example, scaffolds a gradual move from hands-on materials to pictorial representations and then to abstract symbols. Witzel et al. (2003) show how CRA supports building robust foundation of abstract symbols, providing multiple entry points to communicate mathematical concepts. This stepwise progression deepens understanding of mathematical relationships and operations, making it easier to grasp and retain complex mathematical ideas.

While the importance of using realistic representations for low-achieving students is relatively well-acknowledged, less is known about how high-achieving students interact with such representations and how teachers view such interactions. Rau (2017) and others (Dreher & Kuntze, 2015; Stylianou, 2010) suggest that exposure to and work with representations can strengthen high-achieving student critical thinking, problem-solving, and appreciation of mathematical interconnectedness and application in real-world situations. At times, for high-achieving students, the direction may reverse the CRA sequence: starting from abstract concepts and then connecting to realistic contexts to deepen understanding and explore real-world relevance. This can help such learners appreciate the practical utility of mathematical knowledge and develop more flexible thinking about when and how to apply mathematical concepts in various contexts.

The potential ascribed to using realistic representations for supporting both low- and high-achieving students can only be realized if mathematics teachers appreciate and apply multiple representations in their classrooms. However, researchers pointed out that teachers often avoid such use. Dreher and Kuntze (2015) argue that multiple representations are essential for building conceptual understanding, yet the associated interpretive and coordinative demands can overwhelm learners. This dilemma shapes decisions

about when and how to deploy representations, especially in heterogeneous classes; some teachers thus hesitate to introduce multiple forms to low-achieving students, based on their concern that this may confuse students (Stylianou, 2010) and "lead to excessive demands for learners, which may result in confusion and frustration" (Dreher & Kuntze, 2015, p. 27).

Taylor and Dyer (2014) show how teachers navigate these challenges in practice. Their study reveals that teachers' decisions about using mathematical representations are influenced by multiple, often competing goals: supporting conceptual understanding, facilitating mathematical practices, and addressing diverse student needs. A central dilemma identified in their work is the tension between student autonomy—allowing students to choose their preferred representations—and ensuring adequate exposure to a variety of representational forms. Taylor and Dyer (2014) document several specific practices that teachers employ to address this dilemma. Some teachers strategically expose students to multiple representations before allowing them to choose, while others assign specific representations initially to guide learning before encouraging exploration of alternatives. Common practices include requiring students to use at least one additional representation beyond their preferred one to encourage flexibility, facilitating classroom sharing of different representations to promote awareness of multiple modes of thinking. The study also highlights the reflective nature of teachers' practice with representations. This includes responding to student resistance when students are reluctant to adopt unfamiliar representations, as well as developing understanding of how different representations offer distinct affordances for conceptual understanding, procedural fluency, or supporting particular student groups. Specifically regarding realistic representations, some teachers further regard concrete and pictorial representations as temporary scaffolds mainly for low-achieving students, expecting advanced students to work directly with abstract symbols (Moyer, 2001; Zbiek & Conner, 2006). This stance contributes to "fade-out" practices in which realistic representations are used to motivate or illustrate rather than to support deep mathematical work (Fyfe et al., 2014; Witzel et al., 2003).

Teachers' representational choices are often enacted through the tasks they use in their classrooms. Mathematical tasks serve as fundamental sources for creating learning opportunities in mathematics classrooms (Hiebert & Wearne, 1993; Simon & Tzur, 2012; Watson & Ohtani, 2015). Thus, understanding how realistic representations function within these tasks becomes important for effective teaching, particularly in heterogeneous settings. Various frameworks have been developed to analyze mathematical tasks, examining different dimensions such as cognitive demand (Stein et al., 1996), multiple solutions (Leikin, 2011), opportunities for reasoning and proving (Weingarden and Buchbinder, 2025), and inclusive design for diverse learners (Bardy et al., 2021). While some frameworks incorporate representations as one component among broader task characteristics (e.g., Eli et al., 2011; Stein et al., 1996), and others address accessibility for different student populations (e.g., Bardy et al., 2021; Kachwalla, 2021), existing analytical tools provide limited insight into the specific roles that realistic representations play in supporting students across different achievement levels within individual tasks.

Building on this gap, the present study takes an initial step toward investigating the roles realistic representations can play in mathematical tasks for heterogeneous classrooms and how teachers perceive those roles in practice as supporting students across different achievement levels. The commognitive framework guides our conceptualization and analysis of realistic representations as communicational bridges between everyday and mathematical discourses, setting up the theoretical perspective that follows.

3. Theoretical perspective

3.1. Context-based visual mediators: a commognitive perspective

The commognitive framework (Sfard, 2008) conceptualizes thinking as a form of communication and learning mathematics as a change in discourse. Discourse is characterized by participants' ways of saying and doing around mathematical objects, and following Sfard (2007) involves: (1) specialized *keywords*—words signifying quantity, shape, and relationships such as numbers, geometric objects, equality, and similarity; (2) *narratives*—"any text, spoken or written, that is framed as a description of objects, or of relations between objects or activities with or by objects, and that is subject to endorsement or rejection, that is, to being labeled as true or false" (p. 572); (3) discursive *routines*—"well-defined repetitive patterns in interlocutors' actions" (p. 572) such as calculating or solving an equation; and (4) *visual mediators*—"means with which participants of discourses identify the object of their talk and coordinate their communication" (p. 573). Visual mediators include digits, algebraic notations, graphs, geometric drawings, and diagrams.

Expanding this view beyond mathematical objects, visual mediators can also coordinate discourse about everyday situations—for example, a child cycling on a mountain, a basketball shot, a balanced scale, or an elevator moving between floors (Fig. 1). These mediators can help participants coordinate their communication about different situations, not necessarily about mathematical objects, enabling participants of varying mathematical proficiencies to participate in the discourse. Although such mediators may involve mathematical notations (e.g., numbers), "the object of their talk" (Sfard, 2007, p. 573) may be non-mathematical (e.g., mall, elevator, weights). However, they can also be used as a sort of mediator in mathematical discourse. In mathematics teaching and learning, such visual mediators may function as realistic representations. It is important to mention that we are aware that the term "representation" as a construct is not fully compatible with commognition. However, as discussed in the previous section, one way to view representation is as a communication tool. This view of representations as communication tools aligns with the commognitive notion of "mediation", which is the framework we adopt in our study. Accordingly, we have a case for using the commognitive conceptual apparatus in the data analysis.

We propose the term *Context-based Visual Mediators (CVMs)* to examine realistic representations as communicational tools in both everyday and mathematical discourse. For example, an image of an elevator in a shopping mall (cf. Fig. 1) can function as a CVM: students discuss everyday actions like "going up three floors" or "going down to the parking floor" while simultaneously engaging with mathematical concepts such as positive and negative integers and their operations (e.g., $4 - 6 = -2$). The varied ways such

representations mediate across discourse types constitute the focus of our investigation. We are especially interested in the possible roles of CVMs in facilitating transition between everyday and mathematical discourse, acting as interfaces that help students move between familiar, visual, or context-embedded situations and more abstract, symbolic, or formalized mathematical concepts. By exploring the visual and context-based elements of realistic representations as mediators, we aim to clarify their multifaceted contributions to mathematical learning. Our terminology reflects several key considerations consistent with the commognitive perspective: (1) whereas "representations" often implies a direct correspondence to established mathematical objects (i.e., "a representation of an object"), the term "mediators" captures a broader set of roles that context-based visual elements play in mathematical discourse; (2) the visual and context-based elements may mediate everyday discourse before they being associated with specific mathematical objects—supporting communication even when ideas are not yet fully objectified[1]; (3) "mediators" highlights their dynamic, discursive function in enabling communication between discourse realms and supporting transitions between them.

In this exploratory study, we conduct a two-phase investigation. First, we perform an a priori analysis of two representation-oriented activities and their lesson plan intended for heterogeneous classrooms to identify the bidirectional roles of realistic representations as CVMs—that is, how they mediate from everyday to mathematical discourse and vice versa. Second, based on these identified roles, we analyze teachers' discourse to examine how these roles appear in their reflections on implementing the activities in heterogeneous classrooms and whether they view them as supporting (or not) student learning. The research questions that guided this study are:

1. *What are the roles of realistic representations as CVMs in mediating between everyday and mathematical discourses, as identified through analysis of lesson plans of activities involving such representations?*
2. *How do teachers perceive and describe these roles in supporting diverse student learning when reflecting on their implementation of these activities?*

4. Method

This study consists of two complementary parts corresponding to our research questions. First, we conduct an a priori analysis of two activities with realistic representations that were presented to teachers in a professional development program (and later implemented in their classrooms), identifying four distinct roles of realistic representations as CVMs. Second, we use these roles as an analytical framework to examine teachers' discourse about implementing these activities and their perspectives on how realistic representations contribute (or not) to student learning. It is important to note that the teachers were not aware of these four roles during implementation or reflective discussions; rather, the roles served as a methodological tool for analyzing their reflections on the use of realistic representations in heterogeneous classroom contexts.

4.1. Context and participants: professional development for teaching in heterogeneous classrooms

This study was conducted within a Research-Practice Partnership project—Heterogeneous Teaching of Mathematics—aimed at collaboratively developing pedagogies for teaching mathematics in de-tracked, heterogeneous middle-school classrooms in Israel through a partnership between teachers and researchers. The project's approach draws on principles of ambitious teaching, emphasizing student engagement with exploring mathematical ideas, implementing high-cognitive demand tasks, connecting representations to students' everyday experiences, promoting collaborative problem solving, and facilitating discussions in which students share diverse ways of thinking (Casa et al., 2024; Lampert et al., 2010). The project addressed teachers' specific challenges and problems of practice: teachers experienced activities as learners, adapted them to their classroom contexts, implemented them with students, and reflected on their effectiveness to refine pedagogy and determine what worked well and what should be modified (Babichenko et al., 2025).

In 2023–2024, three schools participated in the project, seeking alternatives to the typical tracking of 7th-grade (ages 12–13) mathematics into high- and low-achievement groups. In each school, 7th-grade mathematics teachers participated in Professional Learning Communities (PLCs) and met with two project researchers (including the first author) every three weeks throughout the year. The PLCs focused on discussing challenges of teaching heterogeneous classes, introducing effective practices, engaging teachers in activities designed to encourage student participation, and collaboratively planning lessons. During PLC meetings, researchers first had teachers experience the activity as learners (solving the problem, working in groups, and participating in researcher-facilitated mathematical discussion), followed by discussion of pedagogical implementation and potential adaptations for different classroom contexts. Activities were selected from educational websites and task collections designed to engage all students across achievement levels and aligned with ambitious teaching principles.

This study focused on two activities presented during the PLCs of two participating schools—the Going-Up activity and the Hangers Activity. These activities formed the basis of the a priori analysis for the first research question and are described in detail later in this section. At School A, six teachers participated; at School B, eight. The Going-Up activity was introduced at School A's third PLC meeting and School B's fourth; the Hangers Activity was presented at School A's fourth PLC meeting and School B's fifth. Although teachers were encouraged to implement the activities and share reflections in the PLCs, only a small number did so—expected in the

[1] Objectification, as defined by Sfard (2008), is the process through which abstract nouns begin to be used in discourse as if they signify discourse-independent objects. It involves the gradual transition from talking about processes to talking about objects.

project's first year, when trust-building and reservations about classroom observation were salient. In this exploratory study, we therefore examine two cases—Tamara and Iris—who implemented some of the activities in their classes, supplying the data for our second research question.

Tamara, with more than 15 years of teaching experience, taught a heterogeneous 7th-grade class at School A—a secondary school serving learners from diverse socio-economic backgrounds. During the year the data were collected, she served as Head of the middle-school Mathematics Department (grades 7–9), bringing both classroom experience and a departmental leadership perspective to the project. As part of the project, School A had transitioned from ability grouping to heterogeneous classes, aligning with her interest in supporting diverse learners.

Iris, like Tamara, has many years of elementary-school teaching experience and moved to middle-school teaching three years before data collection. She has taught at School B for the past three years. The school serves learners from relatively low socio-economic backgrounds. She teaches a 7th-grade MOFET[2] class—typically designed for high-achieving students as part of a national effort to provide enriched, challenging opportunities to students from diverse backgrounds—and also coordinates the MOFET program at her school. Her strong belief in high-level learning for all students leads her to balance high academic expectations with accessibility for diverse learners.

4.2. Data collection

The data include three components: (1) the original activities and their full lesson plans as they appeared on their source websites, which we used for the a priori analysis addressing the first research question (analyzing the potential roles of realistic representations as CVMs within instructional activities); (2) one classroom lesson observed by the research team, who took detailed field notes during a teacher's implementation of one activity[3]; and (3) teachers' post-lesson debriefing conversations about implementing the activities, audio-recorded and transcribed. Components (2) and (3) served as data for the second research question. Below, we elaborate on each of these components.

4.2.1. The activities

The data for this article pertain to two activities presented to teachers in the PLCs—the Going-Up Activity and the Hangers Activity. These were selected for analysis because they explicitly incorporated realistic representations drawn from everyday contexts in varied ways. While other activities in the project addressed different types of representations (e.g., diagrams, tables), or other aspects of ambitious teaching beyond representations (e.g., exploration, group work), these two uniquely feature the realistic, everyday-context representations central to our exploration.

The Going-Up Activity focuses on representing, describing, and solving word problems involving integer addition and subtraction. It is sourced from the Make Math Moments website (https://learn.makemathmoments.com/task/going-up/), an online resource created by mathematics educators to support instructional practices. The Make Math Moments approach emphasizes creating engaging, conceptual "math moments" that spark curiosity, use visual models, foster meaningful classroom discourse to deepen understanding, and provide opportunities for all students to participate in the mathematical activity. In the Going-Up Activity, a multi-level parking garage serves as a realistic representation to support students' exploration of signed numbers, particularly integer addition and subtraction (see Figs. 3–4 and Table 3). This representation mediates between students' everyday discourse and mathematical discourse.

The Hangers Activity introduces the concept of equations and deepens students' understanding of equivalency through realistic representations—hangers and scales (see Figs. 6–9). It is sourced from *Illustrative Mathematics* (https://curriculum.illustrativemathematics.org/MS/teachers/2/6/7/index.html), a nonprofit organization founded on the principle that all students can master grade-level mathematics. This principle underpins a comprehensive, problem-based K–12 curriculum designed to empower classrooms and equip students with enduring skills, understandings, and practices that have long-term benefits.

The Going-Up Activity and the Hangers Activity were carefully selected for presentation in the PLC meetings. This choice was guided by pedagogical principles at the core of both activities, aligned with the project's vision and with teaching practices that foster the inclusive participation of all students. In particular, the activities were selected since they utilize real-world, everyday-life realistic representations that are accessible to all students. The activities and their full lesson plans, drawn from the resource materials, were translated into Hebrew by the research team to ensure accessibility for teachers and were then introduced to teachers in PLC meetings.

4.2.2. Lesson observations and post-lesson debriefing conversations

As mentioned above, teachers were asked to implement these activities in their classrooms. The full intended process, from the teachers' perspective, included: (a) experiencing the activity in the PLC as learners; (b) implementing it in class while being observed by the research team, who took detailed field notes; and (c) participating in post-lesson debriefing conversations, which were audio-

[2] MOFET (the acronym for Mathematics, Physics, and Culture in Hebrew) is a non-profit public institution, that promotes academic excellence with emphasis on mathematics, sciences, and languages, and additionally aims at developing personal and social excellence to cultivate a value-driven community. The MOFET network supports designated classes in about 120 middle and high schools all over the country, which are named "MOFET classes."

[3] As described in the methodology section, while additional classroom observations were initially planned, the final dataset included only one classroom observation of one teacher's implementation, leading us to focus primarily on teachers' reflective discourse.

recorded and transcribed. However, due to various constraints—including concerns about being observed and scheduling difficulties—only one teacher (Tamara) allowed classroom observation during the implementation of one activity. Additionally, not all teachers implemented every activity in their classrooms (even without research team observation), as in Iris's case. Table 1 summarizes which parts of the process each teacher completed for each activity.

Overall, the data for this exploratory study includes two activities and their full lesson plans, one lesson observation and three post-lesson debriefing conversations.

4.3. Data analysis

Our analysis followed two main stages using qualitative analysis methods (Miles et al., 2014). The first stage aimed to answer the first research question, which focuses on the roles of realistic representations as CVMs as they appeared in instructional resources for heterogeneous classrooms, mediating between real-world and mathematical discourses. We conducted an a priori content analysis of the original lesson plans of the two activities, combining deductive and inductive elements, to identify the realistic representations and their roles as CVMs. Our analysis focused on the lesson plans' descriptions of anticipated student responses to different parts of the activities, with each part featuring distinct types of realistic representations. We identified anticipated responses and analyzed them according to the characteristics of the discourse (Sfard, 2007), differentiating between everyday and mathematical discourse elements. Regarding the routines students were expected to perform, whereas a routine like "remove a weight of one unit from each side [of the scale]" can be viewed as belonging to everyday discourse, "multiplying each side [of the equation] by 1/3" can be viewed as belonging to mathematical discourse. Similar to routines, also the narratives students were expected to produce can be categorized as belong to everyday discourse or mathematical discourse (e.g., everyday discourse narrative: "the hanger shows a total weight of seven units on one side…" and mathematical discourse narrative: "$7 = 3x + 1$"). Also keywords used in each sub-activity were characterized by their alignment with the two discourses (e.g., everyday discourse keywords: "three floors," "ground level," and mathematical discourse keywords: "negative integer," "absolute value"). Visual mediators were also characterized at the same way: everyday discourse visual mediators included realistic images such as pictures of scales with blocks or parking garage floors, while mathematical discourse visual mediators included symbolic equations or numerical expressions (e.g., $(-2) + (-4) = (-2) - 4$). Then, we systematically analyzed all realistic representations appearing in the activities. For each realistic representation, we examined the specific routines, narratives, and keywords that appeared in its context within the lesson plans. This contextual analysis enabled us to map how each representation mediated discourse—whether it facilitated routines, narratives, and keywords within everyday discourse, within mathematical discourse, or served as a bridge between the two discourses. The systematic comparison of these contextual patterns across all realistic representations revealed four distinct ways in which realistic representations function as CVMs, resulting in the identification of four possible roles of realistic representations as CVMs (as described below in the Findings section). These four identified roles were then examined in two contexts: Tamara's classroom implementation of the Hanger Activity and the post-lesson debriefing conversations with Tamara and Iris.

The second stage addressed the second research question, which examines how teachers viewed the different roles of realistic representations as CVMs in supporting or hindering learning across different achievement levels. This stage involved three phases: (1) using the four CVM roles identified earlier as an analytical framework and examining whether and how teachers referenced them; (2) distinguishing which student groups (low-achieving, high-achieving, or all) teachers associated with each role, and (3) identifying positive or negative evaluative expressions regarding CVMs' contribution (or lack thereof) to student learning.

In the first phase, we coded teachers' statements according to the four identified CVMs roles of realistic representations. For

Table 1
Data collection means completed for Tamara and Iris for each of the two activities.

	Going-Up Activity		**Hangers Activity**	
	Tamara	**Iris**	**Tamara**	**Iris**
Experiencing the activity as learners during the PLC	Did not experience the activity as a learner, but was introduced to the lesson plan suggested for the activity, including its main ideas, goals and practices.	Experienced the activity as a learner in the PLC meeting.	Experienced the activity as a learner in the PLC meeting.	Experienced the activity as a learner in the PLC meeting.
Implementing the activity in class	Implemented the activity in her heterogeneous class. The research team was not present in this lesson.	Did not implement this activity in her class.	Implemented the activity in her heterogeneous class. The research team was invited to observe the lesson. Field notes were taken.	Implemented the activity in her MOFET class. The research team was not present in this lesson.
Post-lesson debriefing conversation	Conducted several days after the lesson. Tamara reflected on the implementation of the activity and shared challenges and dilemmas she encountered.	N/A	Conducted several days after the lesson, in the form of a semi-structured interview[a]	Conducted several days after the lesson. Iris reflected on the implementation of the activity and shared her view about how the activity supported students' learning.

[a] The interview included questions related to teachers' reflections on the lesson implementation, focusing on their overall impressions of how the lesson went, specific successes and challenges they encountered, and their observations about how different students in their class engaged with and learned from the activity.

example, when a teacher described that a particular part of the activity, containing a specific realistic representation, was helpful because it created motivation among students, we categorized this under the motivating role. When a teacher discussed how students made extensive connections between the visual representation (e.g., the scale) and symbolic representation (e.g., $7 = 3x + 1$), we categorized this under the connecting role. Similarly, statements about representations helping students get started with problems were coded under the initiating role, and statements about applying mathematical concepts to real-world contexts were coded under the applying role.

In the second phase, we identified whether teachers were referring to high-achieving students, low-achieving students, or all the students (without explicit differentiation between low- and high-achieving students). Sometimes this was stated explicitly; at other times it was implicit in phrases such as "students who struggle more," "students who always finish first," or "students who always get hundreds." For the MOFET class—typically intended for high-achieving students, as described in the context and participants section (Iris's classroom)—we inferred that all of the teacher's statements referred to high-achieving students.

In the third phase, we identified statements indicating teachers' positive or negative evaluation of using realistic representations, sometimes explicit and sometimes implicit. For example, positive evaluations included statements such as "it helped everyone participate in the lesson and show interest," whereas negative evaluations included statements such as "it bored them and they wanted to move on already."

5. Findings

5.1. Roles of realistic representations as CVMs in the lesson plans of the two activities

Here, we present our a priori analysis of the two activities. Although the activities were designed for heterogeneous classrooms and introduced to teachers during the professional development project, our analysis focuses on identifying the roles of realistic representations themselves rather than their specific application in heterogeneous contexts. We develop the explicit connection between these roles and their potential to support different students in heterogeneous classrooms in the second section of the findings and in the discussion.

The analysis of the two activities revealed four roles of realistic representations as CVMs. The description of these four roles—motivating, initiating, connecting, and applying, appears in Table 2.

Fig. 2 describes how everyday and mathematical discourses are intertwined in each CVM's role.

We now turn to detailing the manifestation of the four roles in each of the two activities.

5.1.1. The Going-Up Activity

The parking representation as a mediator in everyday discourse, motivating student participation. The activity starts with students viewing a brief video featuring a neighborhood with a prominent parking garage (see Fig. 3). Students are then prompted with an open-ended question: "What do you notice, and what do you wonder?".

In this initial phase, the garage building, as depicted in Fig. 3, functions as a visual mediator within everyday discourse, primarily evoking producing narratives about familiar concepts such as cars, parking, streets, and urban structures. The lesson plan provides examples of anticipated student responses. These responses demonstrate that at this stage, students' narratives are not centered on mathematical objects and do not exhibit characteristics of mathematical discourse. Examples, taken from the lesson plan, include:

- "I notice a street with buildings."
- "I notice a large building in the middle."
- "I notice there is a car in the building."
- "I wonder if it is a parking garage."
- "I notice there are twelve levels."
- "I wonder why there is a car inside."
- "I notice it says ground level."

Table 2
Four roles of realistic representations as CVMs.

Motivating	The realistic representation is used solely within the real-world, everyday discourse, typically excluding mathematical discourse characteristics. While a specific mathematical idea is mentioned in students' discourse with the realistic representation, the representation itself remains part of everyday discourse.
Initiating	The realistic representation supports students in entering a new discourse about a potentially new mathematical object. This role involves some characteristics of mathematical discourse while maintaining strong foundations in everyday discourse. Interactions with realistic representations as initiating include keywords and narratives that can belong to both mathematical and everyday discourses.
Connecting	This role involves explicit connections between mathematical discourse characteristics and everyday discourse characteristics. The representation explicates the reasoning behind mathematical processes and can serve as a problem-solving tool alongside or in place of "pure" mathematical procedures. Interactions with realistic representations as connecting include bidirectional shifts between real-world, everyday discourse and mathematical discourse.
Applying	The realistic representation serves more as a background to mathematical notations. Students primarily interact within the mathematical discourse, with the realistic representation providing opportunities to apply mathematical narratives to everyday situations from real-world discourse.

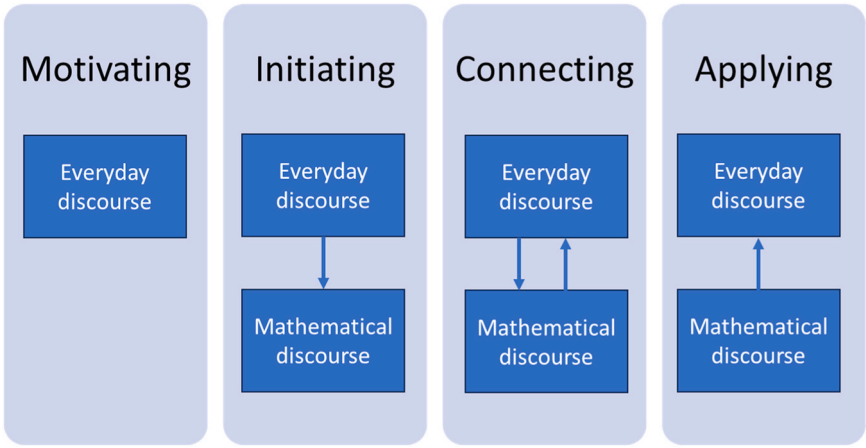


Fig. 2. Relationship between everyday and mathematical discourses in each of the four roles.



Fig. 3. The realistic representation as motivating student participation.

These responses indicate that students initially engage with the realistic representation through everyday discourse. They attend to observable features of the parking garage and its surroundings, without yet invoking mathematical keywords, narratives, routines, or visual mediators (e.g., notations). This introduction invites participation from all students, regardless of mathematical proficiency, thereby supporting the inclusive nature of the activity in heterogeneous classrooms. Thus, in the activity's introduction (Fig. 3), the realistic representation serves to motivate participation, with the visual mediator situated entirely within the realm of everyday experience.

The parking representation as initiating students into mathematical discourse. The second phase of the activity introduces a more detailed visual mediator of the parking garage, accompanied by the prompt: "Each floor in this parking lot is labeled to ensure that all customers can remember where they parked their car. How do you think floors are labeled? Justify your answer." At this stage, the activity and anticipated responses remain primarily within everyday discourse. While the goal is to guide students' thinking towards a floor labeling system relative to the ground level, explicit mathematical objects are not yet introduced. The parking-garage image continues to function mainly as a visual mediator in everyday discourse, but it implicitly begins to plant the initial seeds to

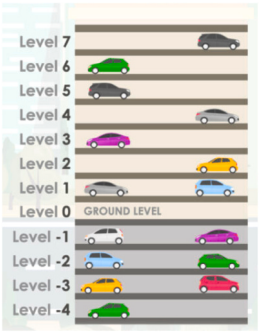


Fig. 4. The realistic representation as initiating students into mathematical discourse.

encourage students' participation in mathematical discourse—bridging familiar contexts (floor labeling) and mathematical concepts (negative integers and absolute value).

The activity reaches a pivotal stage when students are shown the actual floor labels of the parking garage (Fig. 4). This marks a key transition in the role of the visual mediator, as it explicitly begins to function also as a visual mediator in mathematical discourse, incorporating mathematical symbols. In this form, it invites discussion of integers—especially negative numbers, their opposites, and absolute value. The representation reinforces the idea that the ground level corresponds to zero, with all other floors (integers) represented on a number line showing their relative distance from this reference point. By leveraging the realistic representation of the parking garage as a mediator in the mathematical discourse in the form of a number line, the activity supports a smooth shift from everyday context to mathematical discourse, potentially encouraging students to produce narratives about integers and real-world applications while engaging in routines such as comparing and adding.

The parking representation as provoking connections between discourses. The fourth segment of the activity presents four scenarios describing vertical movements of a car within the parking garage. Students use the visual mediator and formulate an equation for each scenario (see Table 3). This phase marks a further shift toward formal mathematical discourse while maintaining connections to the real-world context. Examples of the scenarios are presented in Table 3.

These scenarios use the parking garage to connect everyday experiences with abstract mathematical concepts. By asking students to use both the visual model and equations, this part of the activity facilitates shifts between everyday, context-based narratives (e.g., "the car is moved three levels down", scenario 3), and mathematical narratives (e.g., " $-3+6=3$", scenario 1). Students likewise engage in different types of routines—from everyday actions such as moving the car down three floors to mathematical routines such as subtracting 3 from the current position or adding -3. The keywords shift accordingly, from context-specific terms like "ground level" and "above/below ground" to mathematical terminology such as "negative integers" and "positive values." This multilayered transition across visual mediators (equation and the parking-garage image), narratives, routines, and keywords provides students with opportunities to connect the two discourses.

The parking representation as eliciting application of mathematics in everyday discourse. In this part, students are asked to: "Write two equations that would result in a negative integer. Write a context for each equation." Here, the initial prompt involves only mathematics-related characteristics: mathematical keywords ("equations," "negative integers"); mathematical routines (writing an equation); mathematical mediators (symbolic notation); and mathematical narratives (equations that result in a negative integer). Unlike earlier parts, the realistic representation is not invoked in the prompt. The second part of this task is to write a context for the equation. Although the prompt does not explicitly require using the parking-garage representation, students first produce mathematical narratives (equations) and then everyday narratives. These everyday narratives may draw on the realistic representation, taking into account that this is the "context" that students encountered in previous parts of the activity.

Fig. 5 summarizes the four roles of realistic representations as CVMs identified in the Going-Up lesson plan by examining how everyday and mathematical discourses intertwine. It shows how the mathematical elements of the realistic representation become increasingly salient across the different roles, tracing the progression of student interaction with the realistic representation from primarily everyday discourse to more mathematically oriented engagement. The figure also highlights that everyday discourse remains present even as mathematical keywords, narratives, mediators, and routines become dominant, underscoring the continued interplay between real-world context and abstract mathematical ideas throughout the activity.

We now turn our attention to the Hangers Activity, where similar roles were identified.

5.1.2. The Hangers Activity

The four roles of realistic representations as CVMs, identified in the Going-Up Activity, are also evident in the second activity used in the PLC meetings. We next describe this activity and show how these roles are incorporated into its lesson plan.

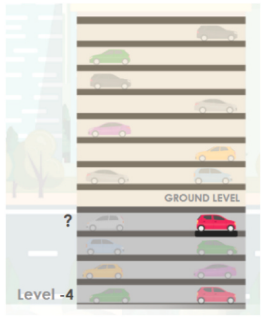
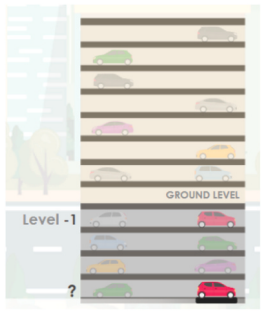
The hangers representation as a mediator in everyday discourse, motivating student participation. The Hangers Activity begins with an image of hangers laden with socks (Fig. 6). As in the Going-Up Activity, students are prompted with two open-ended questions: "What do you notice? What do you wonder?" This introductory phase—parallel to the motivating role of the parking representation—aims to invite participation from all students in a heterogeneous classroom, regardless of mathematical proficiency or prior knowledge. The hanger functions as a realistic representation situated in everyday discourse about familiar objects (hangers and socks), without explicitly invoking characteristics of mathematical discourse.

As indicated by the anticipated responses in the lesson plan, this initial engagement invites everyday, real-world, context-oriented conversation in which mathematics is not yet the primary focus. Students are encouraged to discuss concepts such as "heavier than," "balanced," and "unbalanced"—terms drawn from everyday discourse. Anticipated student narratives from the lesson plan include:

- "There are two pink socks and two blue socks."
- "The socks are clipped to either end of two clothes hangers. The hangers are hanging from a rod."
- "The hanger holding the pink socks is level; the hanger holding the blue socks is not level."
- "Why is the hanger holding the blue socks not level?"
- "Is something inside one of the blue socks to make it heavier than the other sock?"

The hangers representation as initiating students into mathematical discourse. The second phase of the activity introduces two diagrams (Fig. 7) and prompts students to identify one thing that must be true, one that could be true, and one that cannot be true. According to the lesson plan, this segment aims to provide students with opportunities to:

Table 3
Scenarios exemplifying the Connecting role of the realistic representation as CVMs.

Scenarios		The matching realistic representation
Scenario 1	The car is parked on the third floor below ground level. The car is moved up six levels. What floor is the car parked on now? Use a model to convince someone of your answer. Write an equation to represent this scenario.	
Scenario 2	The car is parked on the fourth floor below ground level. The car is moved up three levels. What floor is the car parked on now? Use a model to convince someone of your answer. Write an equation to represent this scenario.	
Scenario 3	The car is parked on the first floor below zero. The car is moved three levels down. Where is the car parked now? Use a model to convince someone of your answer. Write an equation to represent this scenario.	

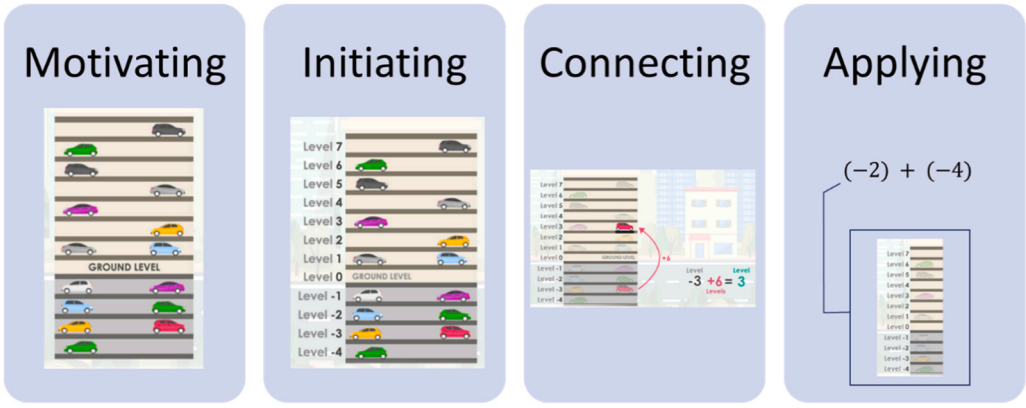


Fig. 5. The four roles of a realistic representation as CVM in the Going-Up Activity.

"Understand how the hanger diagrams work. When the diagram is balanced, there is equal weight on each side. For example, since diagram B is balanced, we know that one triangle weighs the same as three squares and one circle. When the diagram is unbalanced, one side is heavier than the other. For example, since diagram A is unbalanced, we know that one triangle is heavier than one square."



Fig. 6. The realistic representation of hangers laden with socks as motivating student participation.

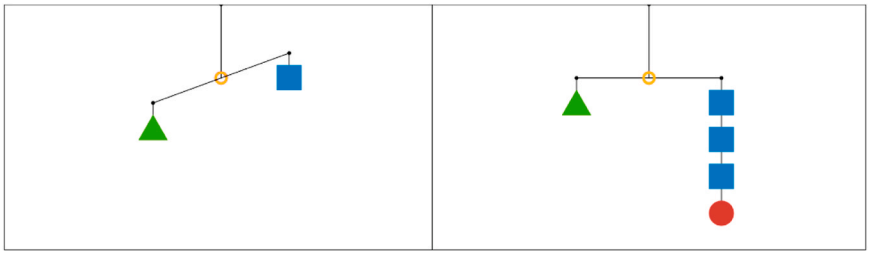


Fig. 7. The realistic representation of diagrams as initiating students into mathematical discourse.

The diagrams and accompanying questions still belong primarily to everyday discourse rather than formal mathematical discourse, as they lack numbers and symbolic notations. Nevertheless, this phase begins to initiate students into mathematical discourse about equations. The keywords shift from purely contextual terms to more mathematically-oriented language—introducing concepts such as "balanced," "equal weight," and "each side," terminology that prepares students for formal equation language. A notable shift from the previous phase concerns the nature of students' narratives: while the sock scenario allowed a wide range of imaginative responses, the diagram imposes definite constraints on possible interpretations. Anticipated narratives evolve from open-ended storytelling to more structured, logical statements (e.g., "one triangle weighs the same as three squares and one circle"), reflecting the transition from everyday to mathematical discourse.

The hangers representation as provoking connections between discourses. In the third phase, students use hanger diagrams to find unknown values and to develop general strategies for solving equations symbolically. As shown in Fig. 8, they match an equation to each of four hanger diagrams. According to the lesson plan, this segment engages students in analyzing relationships and connecting the mathematical narrative (symbolic equations) with the realistic representation of the diagrams. Students use both diagrams and equations to determine the unknown values, explicitly connecting the realistic representation to the mathematical notation of equations.

The lesson plan elaborates on the purpose of the activity, focusing on the realistic representation's role as connecting: "…Students notice the structure of equations and diagrams and find correspondences between them and between solution strategies." It further suggests a teaching strategy (Fig. 9):

> "Invite students to demonstrate, side by side, how they reasoned with both the diagram and the equation. For example, diagram A can be shown next to the equation $7 = 3x + 1$ (see Step I in Fig. 9). Cross out a piece representing 1 from each side, and write $7 - 1 = 3x + 1 - 1$ (Step II), followed by $6 = 3x$ (Step III). Encircle 3 equal groups on each side, and write $6 : 3 = 3x : 3$ (Step IV), followed by $2 = x$ (Step V)…."

This phase facilitates participation in both discourses, with students simultaneously carrying out everyday routines—physically manipulating diagram elements ("cross out a piece," "encircle…")—and mathematical routines—symbolic manipulation and algebraic procedures ("subtract q from each side," "divide each side by p"). The keywords span both discourses, from everyday terms like "equal groups" to mathematical terms such as "equation," "number," and "solution."

The connecting role of the realistic representation is also evident in the two key learning outcomes emphasized by the lesson plan designers:

1. "An instant recognition of the structure of equations of the form $px + q = r$ where p, q, and r are specific, given numbers."
2. "A visual representation in their minds that can be used to support understanding of why for equations of this type, you can subtract q from each side and then divide each side by p to find the solution."

The hangers representation as eliciting application of mathematics in everyday discourse. In the final phase of this activity,

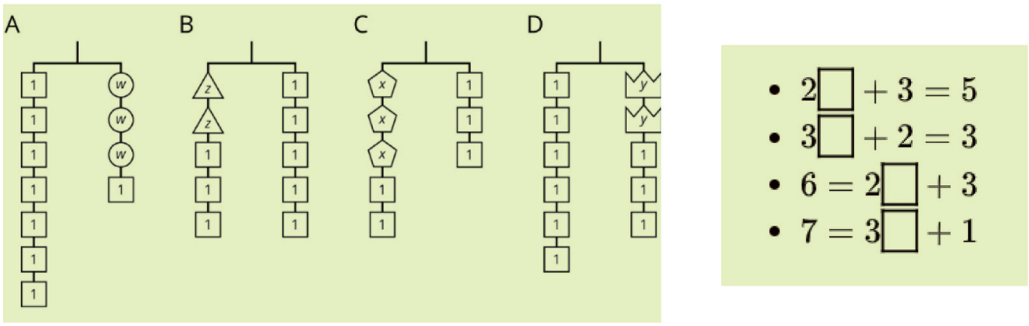


Fig. 8. Diagrams and symbolic narratives exemplifying the connecting role of the realistic representation as CVM.

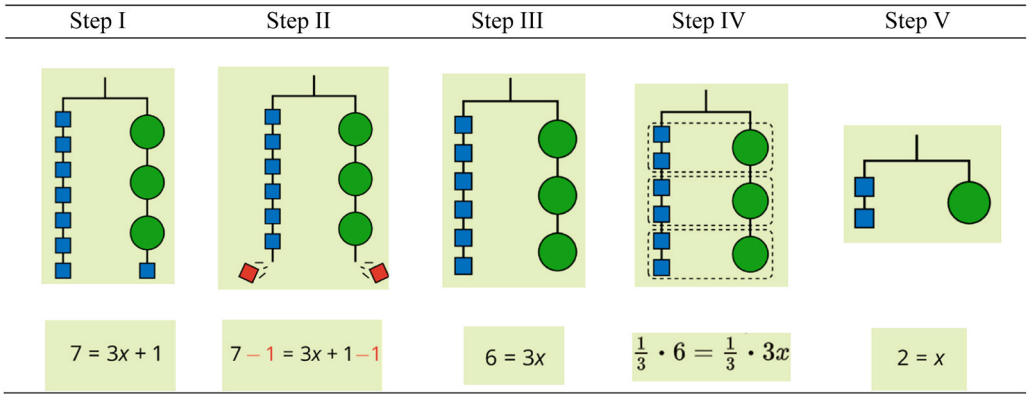


Fig. 9. A possible option of presenting the diagrams and the equations, side by side, as suggested in the lesson plan.

students are presented with an equation: $4x + 6 = 9.2$ and are asked to draw a corresponding hanger diagram for the given equation. This part of the activity, similar to the concluding segment of the Going-Up Activity, uses prompts from mathematical discourse. The aim is to provide students with opportunities to apply the developed mathematical narratives and routines in the form of symbolic equations back to the realistic representation as a mediator in everyday discourse. With the applying role, students are encouraged to use their newly produced mathematical narratives and routines and to translate them into real-world situations using a realistic representation. Fig. 10 shows the four roles as they are manifested in the Hangers Activity.

Thus far, we have examined the design and structure of the Going-Up and the Hangers activities, resulting in four roles of representations as CVMs. We now apply these four identified roles as an analytical framework to examine teachers' reflective discourse following their implementation of these activities, exploring how teachers view and perceive the potential of realistic representations to support diverse learners.

5.2. The representations' roles as supporting student learning: teachers' perspectives

The findings presented here concern how the two teachers, Tamara and Iris, perceived the roles of realistic representations across different student populations as they implemented the activities. By examining teachers' reflections during the post-lesson conversations—and in one case, also the implementation itself—we aim to unpack their perspectives on the roles of realistic representations as CVMs. Specifically, we show how the teachers referred to the four identified roles as supporting (or not) learning for different students, and the extent to which they value (or devalue) the use of such representations.

5.2.1. The motivating and initiating roles

The analysis of Tamara's reflections on her implementation of the Going-Up Activity revealed that she valued the parking-garage realistic representation as motivating—encouraging participation of "all kinds of students," as she said, and creating curiosity at the beginning of the lesson. In her reflection, she noted that when first introduced to the task, students showed engagement and interest; Tamara sensed their unexpressed anticipation, as she said in the debriefing conversation: "Wow, where is this developing to? Where is it going?". However, when she moved to use the realistic representations as initiating—showing how it relates to the number line and to the idea of absolute value—she expressed dissatisfaction ("I found myself quite frustrated"). She linked this to the reactions of students she referred to as "strong": "they immediately understood the idea. As soon as I presented the parking garage [Going-Up Activity], they told me, 'this is the ground, this is zero. This is negative, this is positive' (pointing with her hands)… They

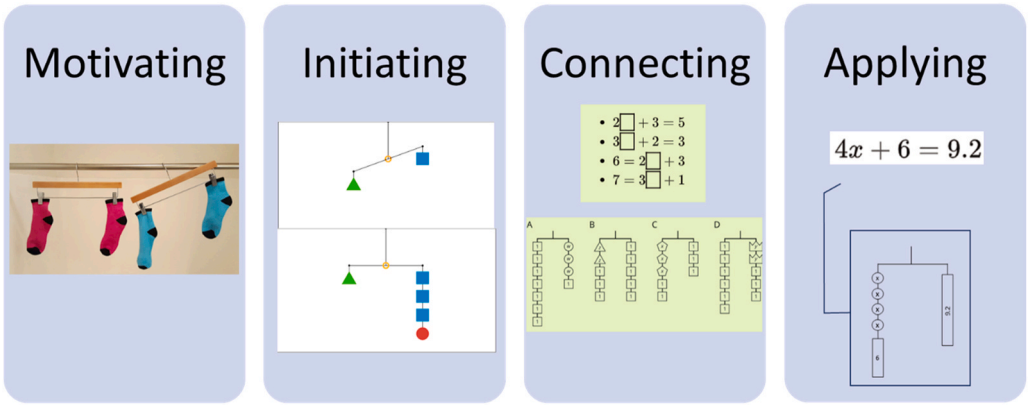


Fig. 10. The four roles of realistic representation as CVM in the Hangers Activity.

understood. And that's what drove them crazy." Thus, she devalued the use of the parking-garage representation as initiating, particularly for supporting high-achieving students. Although she mentioned the potential contribution of the activity to other students (without explicitly detailing this contribution), she was more concerned about the "strong students'" voices in her class:

> "So maybe it [the parking garage image] did serve a few kids who sat in class, the 10 % of the students in [the level of] Group C[4] may have benefited from it. But the rest were bored… They also said that. They were not ashamed to say it… And I left the lesson feeling quite frustrated."

Later in her reflection, Tamara stated that using the realistic representation in its motivating and initiating roles was unnecessary for high achievers because, as she put it, it did not include "real mathematics."

In Tamara's reflection on the Hanger Activity, she discussed the affordances of the hangers representation without tying them to particular student groups (e.g., "strong students," "Group C"). She valued its use for opening a lesson but devalued it as a mediator within mathematical discourse:

> "I think that if I had come in (the classroom) and written an equation on the board and tried to solve it right away… yes. That would already be intimidating. And here, the socks, the hangers… it's something softer. I don't know what to call it. Neutral, maybe… It's a task for an introductory lesson. At some point, you have to engage with the mathematics. There's no way around it."

Following the observation, the researcher shared impressions about how different students participated, which prompted Tamara to elaborate on how using the hangers representation in its motivating and initiating roles can support specific student groups. For example, she described how it helped a particular student:

> "… one of the students … She has good mathematical thinking. But she doesn't have a very high sense of self-efficacy. She sometimes gives very nice answers in various lessons, but she puts herself down… When she gets into it, you can see that she definitely understands. And here [with the Hanger Activity] she did express herself. That means the task might have brought out her sense of self-efficacy because it was something new for everyone. The answers would always be correct]"what do you notice? What do you wonder?" types of questions). The hangers, what are we talking about? What does it mean? What do you think? What do you know for sure? I don't know… Here she participated more than usual… Because it was a task that they [low-achieving students] could handle. It was very visual. They could easily cope with it".

During her observation, the researcher followed a group of students whom Tamara referred to as the "strong students". The researcher noted that, although they solved the equation quite quickly in a symbolic way, matching equations to the hanger images and solving for the variables using the hangers was challenging for them. When asked by the researcher during group work, they struggled to explain their reasoning and solution method. This impression was discussed with Tamara in the debriefing to prompt reflection on how the realistic representation could support these students' learning and productively challenge their mathematical thinking. Tamara was surprised by this observation, particularly by the suggestion that students who solve problems well and quickly may still lack deep understanding of the underlying mathematical ideas. Her perspective (see quotation below) framed the quick completion of tasks as a time-management challenge:

> "My challenge is the strong students in my group, like Ron, whom you saw at the end, and another one, Bar… They are always working together, always finishing my tasks in a minute and a half, and then they lose interest. Then they either start to disrupt or… it's always a challenge. I always think about them. How long will it take them to complete the tasks? Even when I create a

[4] The term "Group C" typically refers to the lowest achievement group when tracking students based on academic performance.

> test or a quiz, I always think about where it will challenge them or how quickly they will finish the task. And somehow, it isn't very good for the other students… so it's a bit stressful. This reality isn't always good."

While Tamara was attuned to the voices of the "strong students" during the lesson—announcing loudly that they finished—she seemed less aware of *how* they engaged with the task. In her reflection, she did not discuss the mathematical affordances of the activity or how the realistic representation supported or hindered these students' learning. Despite the researcher's prompt, suggesting how high achievers might benefit from connecting the symbolic equations and the hangers, Tamara's reflection focused mainly on the *motivating* and *initiating* roles of the representation as a CVM.

In contrast, as we show next, Iris's reflection on her implementation of the Hanger Activity centered on *connecting* and *applying* roles of the hanger representation.

5.2.2. The connecting and applying roles

Iris was very satisfied with how her students engaged with the Hanger Activity:

> "They were so excited and happy in the lesson… They understand the idea. They arrived at the rule (of how to find the unknown amount) totally on their own, and it was so nice to see."

Unlike Tamara, who taught a heterogeneous classroom, Iris taught this lesson in her 7th-grade advanced class. This distinction is important: whereas Tamara devalued the initiating role of the realistic representation for her stronger students, Iris appreciated the activity and how her advanced students connected the hangers to the idea of equation:

> "The word 'equation' was not mentioned in the lesson. I started with the scale, asking what they know about the scale. I didn't refer to the equation. But after we did it, they asked, 'Oh, is this an equation?' They came to the realization that it was an equation. It was so nice to see it. I loved that this happened after they learned the formula for an equation [the symbolic representation]. They saw the equation within the task. They understood it was an equation without me using the term."

Iris's students engaged with the Hanger Activity after learning about equations and how to solve simple equations of the form $ax + b = c$. In her reflection, Iris described the hanger representation as serving the CVM role of applying, emphasizing that it gave students an opportunity to apply mathematical narratives learned in another context ("they came to the realization that it was an equation"), without her stating it explicitly. Her reflection indicates that she values this way of learning ("I loved that this happened"). She also pointed to the CVM role of connecting, valuing the use of the scale to find the unknown value rather than the so-called traditional method they had learned for solving equations:

> "They were truly excited… Even though they knew it was an equation, they kept using the method of the hangers because they were interested in how it would turn out. [In regard to] the new equation they were asked to solve at the end, either as an equation or as a drawing (of a scale), they immediately did it as a drawing and erased the parts on both sides… Then, in the end, after all of that, I asked, 'What equations are these? If you already said it's an equation.' Then above each hanger, they wrote the equation as if saying what the equation was. Not the other way around. They didn't start from the equation."

Iris enjoyed seeing her students choose to find the unknown value by sketching ("drawing") scales and operating with them ("erased the parts on both sides") rather than using the symbolic methods they had learned. Her reflection shows that she values this realistic representation in its connecting and applying roles. In the context of an advanced class, her reflection illustrates that such a representation can be a productive mediator for high-achieving students as well.

In summary, Tamara's reflection focused more on the motivating and initiating roles. She valued realistic representations for creating curiosity, interest and motivation at the beginning of the lesson for "all kinds of students". She also valued them as initiating, but only for low-achieving students ("It was very visual. They could easily cope with it"). With respect to her high-achieving students, she devalued the initiating role because "it did not challenge them". Her devaluation also seemed tied to the absence of explicit, purely mathematical notation ("At some point, you have to engage with the mathematics. There's no way around it"). In contrast, Iris valued the connecting and applying roles, maintaining that realistic representations provide advanced students with opportunities to translate the mathematics they are learning and apply it across models and situations, including those that do not explicitly involve purely mathematical notation.

6. Discussion

This study examined the use of realistic representations in heterogeneous classrooms and their bidirectional role as CVMs (Context-based Visual Mediators), mediating between everyday and mathematical discourses. We identified intended roles through an a priori analysis of two activities and their original lesson plans, and we unpacked teachers' views on how these roles facilitate student learning across different levels of mathematical proficiency. In this section, we discuss our findings in relation to the research questions.

6.1. Roles of realistic representations as bridging real-world and mathematical discourses

The first research question examined the roles of realistic representations as CVMs in mediating between everyday and mathematical discourses, as manifested in the lesson plans of activities involving such representations. Analyzing two activities—the Going-Up Activity and the Hangers Activity—we identified four roles: motivating, initiating, connecting, and applying, and for each, we

articulated the relationship between everyday and mathematical discourse. In what follows, we discuss the affordances of these roles for mathematics teaching and learning.

The motivating role of realistic representations as CVMs, evident in the initial stages of both activities, supports the creation of an inclusive classroom environment. By presenting familiar, real-world contexts without explicit mathematical notation (e.g., a picture of hangers with socks; a parking garage), such representations can encourage participation from all students. This aligns with studies emphasizing the importance of using multiple means of engagement to foster inclusive classrooms and provide learning opportunities for all students (Horn & Garner, 2022; Jackson et al., 2013).

The initiating role of realistic representations as CVMs enables a gradual transition from everyday discourse to mathematical discourse. This scaffolding bridges students' everyday experiences and formal mathematical keywords, narratives, routines, and visual mediators (notations). It aligns with Van den Heuvel-Panhuizen's (2003) finding that contextual problems can be productive entry points to abstract mathematical learning. In both activities, realistic representations offered concrete, visual starting points, allowing students to engage in mathematical routines (solving equations; adding and subtracting integers) through familiar scenarios (balancing scales; car movement) before encountering abstract symbols. This approach echoes Fyfe et al.'s (2014) notion of "concreteness fading," in which learning—and the instruction that supports it—progresses from concrete to abstract representations.

The connecting role of realistic representations as CVMs, evident in the later stages of both activities, helps students make explicit connections between real-world contexts and mathematical objects. In the Going-Up Activity, students connect car movements to integer operations, while in the Hangers Activity, they link "keeping the scale balanced" to equation solving. This connecting role aligns what Hiebert and Carpenter (1992) and others (e.g., Arcavi, 2003; Hatisaru et al., 2024; Rowland et al., 2005; Weingarden, 2024), who emphasize the importance of connecting representations for developing mathematical conceptual understanding.

The applying role of realistic representations as CVMs, observed in the final stages of both activities, is realized when students use the mathematical routines and narratives they have produced—together with the mathematical keywords and notation—in real-world contexts. In the Going-Up Activity, students apply integer operations to parking scenarios; in the Hanger Activity, they translate equation-solving strategies to scale-balance problems. This application role aligns with Freudenthal's (1973) view of mathematics as a human activity, in which learners "mathematize" real-world situations, and exemplifies Gravemeijer's (1999) account of models that begin as representations of concrete situations and evolved into tools for mathematical reasoning.

6.2. *Teacher perspectives on roles of realistic representations as CVMs in heterogeneous classes*

The four roles identified in our analysis of the two activities served as an analytical framework for addressing the second research question, exploring teachers' perspectives on how realistic representations can support student learning in heterogeneous classrooms. Following classroom implementation of the activities, we examined which roles teachers referred to as supporting (or not) the learning of different students.

Our findings reveal contrasting views about the value of realistic representations for high-achieving students. Whereas Tamara worried about the absence of explicit mathematical notation, Iris appreciated how these representations facilitated connections between everyday situations and abstract mathematics. Tamara's focus on the motivating and initiating roles—particularly for lower-achieving students—is a well grounded orientation. As noted earlier, realistic representations have been found beneficial for low-achieving learners (Flores, 2010; Karsenty et al., 2007; Kilpatrick et al., 2001; Otten et al., 2019; Subramaniam, 2019; Xin et al., 2022). However, her surprise at the researcher's observations about how higher-achieving students engaged with the Hangers Activity, and how the realistic representation can deepen their mathematical thinking, suggests that integrating realistic representations into heterogeneous classes may require a shift in orientation—viewing them as supporting various learners, not only the low-achieving ones. Iris's perspective is encouraging in that regard: it challenges the common assumption that visual aids and concrete models are primarily beneficial for low-achieving learners, arguing instead that the connecting and applying roles can support the development of conceptual understanding and mathematical thinking for advanced students as well (Dreher & Kuntze, 2015; Taylor & Dyer, 2014).

6.3. *Theoretical and practical contribution*

This study makes a theoretical contribution by offering an explicit, theory-grounded framework for understanding realistic representations and their roles in supporting mathematical learning as mediators between everyday and mathematical discourses. While existing task-analysis frameworks (e.g., Stein et al., 1996; Thompson et al., 2012) examine dimensions such as cognitive demand and opportunities for reasoning and proving, they do not specifically address how representations function communicatively within individual tasks to support diverse learners in heterogeneous contexts. Our CVM framework addresses this gap by providing a discourse-based lens for analyzing how representations afford varied learning opportunities for students across achievement levels.

Identifying and clearly defining the four roles—motivating, initiating, connecting, and applying—offers educators and researchers shared, theory-based terminology for examining the use of realistic representations in mathematics teaching. Grounded in commognitive theory, with its focus on discourse elements (routines, narratives, keywords, and visual mediators), the framework provides analytical tools that extend beyond cognitive or developmental accounts. By examining how representations mediate transitions between discourses, the framework captures the communicational complexity whereby different students engage with the same representation through different discourse pathways—a dimension especially crucial in heterogeneous settings. The framework also contributes to efforts to establish precise, well-defined pedagogical discourse, facilitating more accurate and accessible discussions of teaching practice (Weingarden & Heyd-Metzuyanim, 2025; Weingarden and Buchbinder, 2023). By providing concrete,

operationalized tools for teacher learning (e.g., lesson planning, reflection on practice, pedagogical conversations), it can support practitioners and researchers in making deliberate, theory-informed decisions about using realistic representations while supporting learning across diverse achievement levels.

Beyond the theoretical contribution of the framework itself and the four roles of realistic representations as CVMs, this study also advances understanding of *how* realistic representations can deepen mathematical understanding for all learners—a persistent concern for researchers and practitioners (Kilpatrick et al., 2001; NCTM, 2000). While there is substantial research on gains for *either* low-achieving students (see above), *or* high-achieving students (e.g., Rau, 2017; Stylianou, 2010; Taylor & Dyer, 2014), less is known about how learning and teaching with multiple representations—specifically realistic representations—can be integrated within heterogeneous learning environments. Our study takes a step toward this goal and suggests several implications for mathematics education practice and research.

One key implication is that realistic representations can be strategically integrated into heterogeneous classrooms to serve multiple roles simultaneously. For students who need additional support in engaging with new mathematical concepts, realistic representations can function primarily in the motivating and initiating roles, providing accessible entry points into unfamiliar mathematical discourse. Concurrently, for students who are more familiar with the mathematical content, the same realistic representations can serve as connecting and applying, deepening their understanding by linking abstract concepts to real-world situations. This dual functionality of realistic representations can potentially address the diverse needs within a single classroom, promoting inclusive and differentiated instruction (e.g., Eikeland & Ohna, 2022; Goyibova et al., 2025; Prast et al., 2018).

However, realizing this potential requires teachers' awareness of the different roles that realistic representations can play in heterogeneous classrooms. The contrast between Tamara's and Iris's reflections on the Hanger Activity reveals differences in awareness of representations' roles as CVMs. Tamara's emphasis on the motivating and initiating roles, coupled with her concern about "real mathematics," frames realistic representations primarily as scaffolds. In contrast, Iris recognized how students used the hangers to explore equations, viewing realistic representations as tools for mathematical thinking and problem-solving. This difference in awareness has implications for implementation: limited awareness may lead to missed opportunities to support diverse learners. To address this, professional development should foreground the various roles and their benefits for all students, challenging the notion that realistic representations are primarily useful for low-achieving learners. Case studies like those of Tamara and Iris can be especially valuable, serving as prompts to surface orientations for teacher discussion and learning. By examining contrasting perspectives and experiences, teachers can develop a more nuanced understanding of how realistic representations can be implemented to support mathematical thinking and problem-solving for all students in heterogeneous classrooms.

Teacher professional development programs could incorporate activities such as analyzing classroom videos or student work to identify how different learners engage with realistic representations in various CVM roles. This approach may help teachers recognize the diverse ways students interact with realistic representations across achievement levels. These programs might also provide opportunities for teachers to design and implement activities that serve multiple CVM roles simultaneously, engaging them in creating more inclusive learning experiences. Exploring strategies for facilitating productive mathematical discussions around realistic representations in heterogeneous settings could likewise be valuable, offering opportunities to consider how such representations deepen conceptual understanding across diverse student groups. By engaging in these forms of professional learning, teachers can develop ways to integrate realistic representations into their practice to support all learners.

Future research could delve deeper into teachers' experiences and development in using realistic representations in heterogeneous mathematics classrooms. Longitudinal studies might examine how teachers' awareness, perspectives, and practices regarding the roles of realistic representations as CVMs evolve over time, particularly in response to targeted professional development. Such research could also explore factors that influence teachers' decisions to incorporate realistic representations into instruction and how they navigate the challenges of using these representations to meet diverse student needs. Additionally, case studies comparing teachers who effectively use realistic representations across all achievement levels with those who limit their use to specific student groups could illuminate productive practices and emerging challenges. Research might further examine how teachers' own engagement with realistic representations—and their flexibility in translating between representations as learners—relates to their implementations of such activities in teaching. The focus on two cases aligns with the exploratory nature of the research questions, which examine how representations serve as mediators rather than investigating frequencies or patterns that would require a larger sample. Our analysis can lay the groundwork for future research in this area.

6.4. Limitations

The contexts in which these teachers worked—Tamara in a heterogeneous classroom and Iris in an advanced class—provide an informative contrast but do not represent the full spectrum of classroom environments. The study also relied heavily on teachers' self-reported reflections and limited classroom observations, which may not capture the full complexity of implementing activities with realistic representations in practice. Moreover, the conversational nature of the interviews, while yielding authentic insights into teachers' immediate reflections, did not systematically probe broader pedagogical goals and instructional challenges, that might deepen understanding of their approaches to using representations. Student perspectives and long-term outcomes were not examined in depth, although the study lays groundwork for future research in these areas. In addition, the focus on specific activities (the Going-Up and Hanger activities), limits generalizability to other types of realistic representations or mathematical topics; still, we believe the findings can serve as exemplars for understanding the use of realistic representations more broadly. Finally, the study was conducted within a particular cultural and educational context, so transferability to other systems may be limited; nevertheless, we suggest that the themes addressed are broadly relevant, and the insights—particularly regarding how realistic representations can support all

students—may be applicable across varied settings, even if specific implementations differ.

7. Conclusion

Despite its limitations, this study offers insights into the use of realistic representations in heterogeneous mathematics classrooms. It provides a framework for future investigations into implementing realistic representations across different mathematical topics and grade levels. It also highlights the need for further research on teachers' orientations and practices regarding realistic representations and their roles as CVMs—particularly through targeted professional development. By recognizing, conceptualizing, and utilizing the multiple roles of realistic representations as CVMs, educators can create more inclusive and effective mathematics learning environments that support the diverse needs of all students in heterogeneous classrooms. This study also advances understanding of how teachers can better recognize and leverage the affordances of realistic representations, potentially enhancing mathematical learning across different achievement levels.

CRediT authorship contribution statement

Merav Weingarden: Writing – original draft, Data curation, Conceptualization. **Ronnie Karsenty:** Writing – review & editing, Supervision. **Boris Koichu:** Writing – review & editing, Supervision.

Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used ChatGPT and Claude in order to improve the readability and language of the manuscript. After using these tools, the authors reviewed and edited the content as needed and took full responsibility for the content of the published article.

Declaration of Competing Interest

The authors have no relevant financial or non-financial interests to disclose.

Acknowledgements

The research is supported by a grant from the Center for the Study of Pedagogy: Research-Practice Partnerships at Ben Gurion University of the Negev and by a fellowship from the Dean of Education of the Weizmann Institute of Science. The opinions expressed herein are those of the authors and do not necessarily reflect the views of the funding agencies. The authors would like to thank Anna Sfard for her valuable suggestions, Miriam Babichenko for her advice and collaborative work, and the anonymous reviewers for their careful reading and constructive feedback on earlier versions of this article.

Data availability

Data will be made available on request.

References

Ainsworth, S. (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and Instruction, 16*(3), 183–198. https://doi.org/10.1016/j.learninstruc.2006.03.001

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*(3), 215–241. https://doi.org/10.1023/A:1024312321077

Babichenko, M., Weingarden, M., & Abu Asaad, I. (2025). Exploring teachers' problems of practice regarding ambitious teaching for struggling students. In A. Rajala, A. Cortez, R. Hofmann, A. Jornet, H. Lotz-Sisitka, & L. Markauskaite (Eds.). Proceedings of the 19th International Conference of the Learning Sciences (ICLS 2025), pp. 2732-2734. International Society of the Learning Sciences. Helsinki, Finland. June 2025.

Baccaglini-Frank, A. (2021). To tell a story, you need a protagonist: How dynamic interactive mediators can fulfill this role and foster explorative participation to mathematical discourse. *Educational Studies in Mathematics, 106*(2), 291–312. https://doi.org/10.1007/s10649-020-10009-w

Bardy, T., Holzäpfel, L., & Leuders, T. (2021). Adaptive tasks as a differentiation strategy in the mathematics classroom: Features from research and teachers' views. *Mathematics Teacher Education and Development, 23*(3), 26–53.

Boaler, J. (2008). Promoting 'relational equity' and high mathematics achievement through an innovative mixed-ability approach. *British Educational Research Journal, 34*(2), 167–194. https://doi.org/10.1080/01411920701532145

Boaler, J. (1993). The role of contexts in the mathematics classroom: Do they make mathematics more "real"? For the Learning of Mathematics, 13(2), 12-17. ⟨https://www.jstor.org/stable/40248079⟩.

Boaler, J., & Sengupta-Irving, T. (2016). The many colors of algebra: The impact of equity focused teaching upon student learning and engagement. *The Journal of Mathematical Behavior, 41*, 179–190. https://doi.org/10.1016/j.jmathb.2015.10.007

Boaler, J., William, D., & Brown, M. (2000). Students' experiences of ability grouping: Disaffection, polarisation and the construction of failure. *British Educational Research Journal, 26*(5), 631–648. https://doi.org/10.1080/713651583

Burris, C. C., & Garrity, D. T. (2008). *Detracking for excellence and equity*. ASCD.

Casa, T. M., Cavanna, J. M., & Pinter, H. H. (2024). Beginning to implement ambitious mathematics instruction: What promising first-year elementary teachers' lessons can teach us. *Investigations in Mathematics Learning, 16*(4), 301–319. https://doi.org/10.1080/19477503.2024.2363711

Dreher, A., & Kuntze, S. (2015). Teachers facing the dilemma of multiple representations being aid and obstacle for learning: Evaluations of tasks and theme-specific noticing. *Journal für Mathematik-Didaktik, 36*(1), 23–44.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25–41). Kluwer. https://doi.org/10.1017/CBO9781139013499.008.
Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1), 103–131.
Duval, R. (2020). Registers of semiotic representation. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 724–727). Springer. https://doi.org/10.1007/978-3-319-77487-9_100033-1.
Eikeland, I., & Ohna, S. E. (2022). Differentiation in education: A configurative review. *Nordic Journal of Studies in Educational Policy, 8*(3), 157–170. https://doi.org/10.1080/20020317.2022.2039351
Eli, J. A., Mohr-schroeder, M. J., & Lee, C. W. (2011). Exploring mathematical connections of prospective middle-grades teachers through card-sorting tasks. *Mathematics Education Research Journal, 23*(3), 297–319. https://doi.org/10.1007/s13394-011-0017-0
Flores, M. M. (2010). Using the concrete-representational-abstract sequence to teach subtraction with regrouping to students at risk for failure. *Remedial and Special Education, 31*(3), 195–207. https://doi.org/10.1177/0741932508327467
Fonger, N. L. (2019). Meaningfulness in representational fluency: An analytic lens for students' creations, interpretations, and connections. *The Journal of Mathematical Behavior, 54*, Article 100678. https://doi.org/10.1016/j.jmathb.2018.10.003
Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht, The Netherlands: D. Reidel Publishing Company.
Fyfe, E. R., McNeil, N. M., Son, J. Y., & Goldstone, R. L. (2014). Concreteness fading in mathematics and science instruction: A systematic review. *Educational Psychology Review, 26*, 9–25. https://doi.org/10.1007/s10648-014-9249-3
Fyfe, E. R., McNeil, N. M., & Borjas, S. (2015). Benefits of "concreteness fading" for children's mathematics understanding. *Learning and Instruction, 35*, 104–120. https://doi.org/10.1016/j.learninstruc.2014.10.004
Gamoran, A. (2004). Classroom organization and instructional quality. In M. C. Wang, & H. J. Walberg (Eds.), *Can unlike students learn together? Grade retention, tracking, and grouping* (pp. 141–155). Greenwich, CT: Information Age Publishing.
Gamoran, A. (2010). Tracking and inequality, New directions for research and practice. In M. W. Apple, S. J. Ball, & L. Armando Gandin (Eds.), The Routledge International Handbook of the Sociology of Education (pp. 213–228).
Gardesten, M., & Palmér, H. (2023). Students' participation in mathematics in inclusive classrooms: A study of the enacted mathematical and relational knowing of teachers. *Mathematical Thinking and Learning, 27*(2), 157–177. https://doi.org/10.1080/10986065.2023.2258485
Gibbs, K., & McKay, L. (2021). Differentiated teaching practices of Australian mainstream classroom teachers: A systematic review and thematic analysis. *International Journal of Educational Research, 109*, Article 101799. https://doi.org/10.1016/j.ijer.2021.101799
Goldin, G. A. (1998). Representational systems, learning, and problem solving in mathematics. *The Journal of Mathematical Behavior, 17*(2), 137–165. https://doi.org/10.1016/S0364-0213(99)80056-1
Goldin, G. A. (2020). Mathematical representations. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 566–572). Springer. https://doi.org/10.1007/978-3-030-15789-0_103.
Goldin, G., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. A. Cuoco, & F. R. Curcio (Eds.), *The role of representation in school mathematics* (pp. 1–23). Boston, MA: NCTM.
Goyibova, N., Muslimov, N., Sabirova, G., Kadirova, N., & Samatova, B. (2025). Differentiation approach in education: Tailoring instruction for diverse learner needs. *MethodsX*. , Article 103163. https://doi.org/10.1016/j.mex.2025.103163
Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155–177. https://doi.org/10.1207/s15327833mtl0102_4
Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39*(1), 111–129. https://doi.org/10.1023/A:1003749919816
Hatisaru, V., Stacey, K., & Star, J. (2024). Mathematical connections in preservice secondary mathematics teachers' solution strategies to algebra problems. *Avances Dèlelőtt InvestigaciőN Eno¯sis EducaciőN Matemática, 25*(25), 33–55. https://doi.org/10.35763/aiem25.6354
Hiebert, J. C., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65–97). New York: Macmillan.
Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *In American Educational Research Journal Summer, 30*(2).
Hodgen, J., Foster, C., Brown, M., & Martin, D. (2024). Low-attaining secondary school mathematics students' perspectives on recommended teaching strategies. *International Journal of Science and Mathematics Education, 22*(6), 1325–1343. https://doi.org/10.1007/s10763-023-10420-8
Horn, I., & Garner, B. (2022). *Teacher learning of ambitious and equitable mathematics instruction: A sociocultural approach*. Routledge.
Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, E. (2013). Exploring relationships between setting up complex tasks and opportunities to learn in concluding whole-class discussions in middle-grades mathematics instruction. *Journal for Research in Mathematics Education, 44*(4), 646–682. https://doi.org/10.5951/jresematheduc.44.4.0646
Jong, R. D., Westerhof, K. J., & Kruiter, J. H. (2004). Empirical evidence of a comprehensive model of school effectiveness: A multilevel study in mathematics in the 1st year of junior general education in the Netherlands. *School effectiveness and School improvement, 15*(1), 3–31. https://doi.org/10.1076/sesi.15.1.3.27490
Kachwalla, B. (2021). Making math accessible to all students: Effective pedagogy? *Journal of Higher Education Theory Practice, 21*(3).
Kaput, J. J., Blanton, M. L., & Moreno, L. (2008). Algebra from a symbolization point of view. In In. J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 19–56). Routledge.
Karsenty, R., Arcavi, A., & Hadas, N. (2007). Exploring informal mathematical products of low achievers at the secondary school level. *The Journal of Mathematical Behavior, 26*(2), 156–177. https://doi.org/10.1016/j.jmathb.2007.05.003
Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academies Press.
Kumar, R. S., Subramaniam, K., & Naik, S. S. (2017). Teachers' construction of meanings of signed quantities and integer operation. *Journal of Mathematics Teacher Education, 20*(6), 557–590. https://doi.org/10.1007/s10857-015-9340-9
Lampert, M., Beasley, H., Ghousseini, H., Kazemi, E., & Franke, M. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. In M. K. Stein, & L. Kucan (Eds.), *Instructional explanations in the disciplines* (pp. 129–141). Boston, MA: Springer. https://doi.org/10.1007/978-1-4419-0594-9_9.
Leikin, R. (2011). Multiple-solution tasks: From a teacher education course to teacher practice. *ZDM – Mathematics Education, 43*(6), 993–1006. https://doi.org/10.1007/s11858-011-0342-5
Lenz, K., Obersteiner, A., & Wittmann, G. (2024). Who benefits most from language-responsive learning materials in mathematics? Investigating differential effects in heterogeneous classrooms. *Educational Studies in Mathematics, 116, 185–211*. https://doi.org/10.1007/s10649-024-10321-9
Lepak, J. R., Wernet, J. L., & Ayieko, R. A. (2018). Capturing and characterizing students' strategic algebraic reasoning through cognitively demanding tasks with focus on representations. *The Journal of Mathematical Behavior, 50*, 57–73. https://doi.org/10.1016/j.jmathb.2018.01.003
Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33–40). Lawrence Erlbaum Associates.
Miles, M. B., Huberman, A. M., & Saldaña, J. (2014). *Qualitative data analysis: A methods sourcebook* (3rd Ed.). Sage.
Moyer, P. S. (2001). Are we having fun yet? How teachers use manipulatives to teach mathematics. *Educational Studies in Mathematics, 47*(2), 175–197. https://doi.org/10.1023/A:1014596316942
Mulkey, L. M., Catsambis, S., Steelman, L. C., & Crain, R. L. (2005). The long-term effects of ability grouping in mathematics: A national investigation. *Social Psychology of Education, 8*, 137–177. https://doi.org/10.1007/s11218-005-4014-6
(National Council of Teachers of Mathematics) NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
Oakes, J. (2005). *Keeping track: How schools structure inequality*. Yale University Press.

Otten, M., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (2019). The balance model for teaching linear equations: A systematic literature review. *International Journal of STEM Education, 6*, 30. https://doi.org/10.1186/s40594-019-0183-2
Prast, E. J., Van de Weijer-Bergsma, E., Kroesbergen, E. H., & Van Luit, J. E. (2018). Differentiated instruction in primary mathematics: Effects of teacher professional development on student achievement. *Learning and Instruction, 54*, 22–34. https://doi.org/10.1016/j.learninstruc.2018.01.009
Rau, M. A. (2017). Conditions for the effectiveness of multiple visual representations in enhancing STEM learning. *Educational Psychology Review, 29*, 717–761. https://doi.org/10.1007/s10648-016-9365-3
Robotti, E., Miragliotta, E., & Bonadiman, C. (2023). From weight to value: Focusing on verbal signs during a digital artifact-mediated algebraic activities. In Thirteenth Congress of the European Society for Research in Mathematics Education (CERME 13) (No. 15). Alfréd Rényi Institute of Mathematics; ERME.
Roth, W. M., & McGinn, M. K. (1998). Inscriptions: Toward a theory of representing as social practice. *Review of Educational Research, 68*(1), 35–59. https://doi.org/10.3102/00346543068001035
Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8*(3), 255–281. https://doi.org/10.1007/s10857-005-0853-5
Rubin, B. C., & Noguera, P. A. (2004). Tracking detracking: Sorting through the dilemmas and possibilities of detracking in practice. *Equity Excellence in Education, 37*(1), 92–101. https://doi.org/10.1080/10665680490422142
Schofield, J. W. (2010). International evidence on ability grouping with curriculum differentiation and the achievement gap in secondary schools. *Teachers College Record, 112*(5), 1492–1528. https://doi.org/10.1177/016146811011200506
Selling, S. K. (2016). Learning to represent, representing to learn. *The Journal of Mathematical Behavior, 41*, 191–209. https://doi.org/10.1016/j.jmathb.2015.10.003
Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The Journal of the Learning Sciences, 16*(4), 565–613. https://doi.org/10.1080/10508400701525253
Sfard, A. (2008). *Thinking as communicating*. New York: Cambridge University Press.
Simon, M. A., & Tzur, R. (2012). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. In D. H. Clements, & J. Sarama (Eds.), *Hypothetical learning trajectories* (pp. 91–104). Routledge.
Slavin, R. E. (1990). Achievement effects of ability grouping in secondary schools: A best evidence synthesis. *Review of Educational Research, 60*(3), 471–499. https://doi.org/10.3102/00346543060003471
Stein, M. K., & Bovalino, J. W. (2001). Reflections on practice: Manipulatives: One piece of the puzzle. *Mathematics Teaching in the Middle School, 6*(6), 356–359. https://doi.org/10.5951/MTMS.6.6.0356
Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455–488. https://doi.org/10.3102/00028312033002455
Stylianou, D. A. (2010). Teachers' conceptions of representation in middle school mathematics. *Journal of Mathematics Teacher Education, 13*, 325–343. https://doi.org/10.1007/s10857-010-9143-y
Subramaniam, K. (2019). Representational coherence in instruction as a means of enhancing students' access to mathematics. In Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 33-52). Pretoria, South Africa: PME.
Swanson, H. L., & Jerman, O. (2006). Math disabilities: A selective meta-analysis of the literature. *Review of Educational Research, 76*(2), 249–274. https://doi.org/10.3102/00346543076002249
Tall, D., & Razali, M. R. (1993). Diagnosing students' difficulties in learning mathematics. *International Journal of Mathematical Education in Science and Technology, 24*(2), 209–222. https://doi.org/10.1080/0020739930240206
Taylor, E. V., & Dyer, E. (2014). Teacher goals and dilemmas in the use of mathematical representations. *Mathematics Teacher Educator, 2*(2), 171–184. https://doi.org/10.5951/mathteaceduc.2.2.0171
Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. *Journal for Research in Mathematics Education, 43*(3), 253–295. https://doi.org/10.5951/jresematheduc.43.3.0253
Tomlinson, C. (2001). *How to differentiate instruction in mixed-ability classrooms* (2nd ed.). Alexandria, VA: Association for Supervision and Curriculum Development.
Tripathi, P. N. (2008). Developing mathematical understanding through multiple representations. *Mathematics Teaching in the Middle School, 13*(8), 438–445. https://doi.org/10.5951/MTMS.13.8.0438
Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*, 9–35. https://doi.org/10.1023/B:EDUC.0000005212.03219.dc
Warren, E., & Cooper, T. J. (2009). Developing mathematics understanding and abstraction: The case of equivalence in the elementary years. *Mathematics Education Research Journal, 21*(2), 76–95. https://doi.org/10.1007/BF03217546
Watson, A., & Ohtani, M. (2015). *Task design in mathematics education: The 22nd ICMI study (New ICMI study series)*. New York: Springer.
Weingarden, M. (2024). The role of mathematical connections in object-level and meta-level learning: A potential lens for supporting pre-service teacher learning. *Research in Mathematics Education, 26*(2), 258–282. https://doi.org/10.1080/14794802.2024.2371542
Weingarden, M., & Buchbinder, O. (2023). Applying a commognitive-based framework to promote teachers' communication about reasoning and proving. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education (PME), (Vol. 4), pp. 339-346. Haifa, Israel. July 2023.
Weingarden, M., & Heyd-Metzuyanim, E. (2021). Opportunities for explorative participation in different achievement groups. In Inprasitha, M., Changsri, N. & Boonsena, N. (Eds.). Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education (pp. 675-683). Khon Kaen, Thailand: PME. July 2021.
Weingarden, M., & Buchbinder, O. (2025). Exploring the opportunities for reasoning and proving in mathematical tasks designed by secondary prospective teachers. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-025-10440-x
Weingarden, M., & Heyd-Metzuyanim, E. (2023). What can the realization tree assessment tool reveal about explorative classroom discussions? *Journal for Research in Mathematics Education, 54*(2), 97–117. https://doi.org/10.5951/jresematheduc-2020-0084
Weingarden, M., & Heyd-Metzuyanim, E. (2025). Fostering pre-service teachers' attention to mathematical objects: The Realization Tree mediator as a teaching representation. *Journal of Mathematics Teacher Education, 28*, 851–877. https://doi.org/10.1007/s10857-024-09622-w
Witzel, B. S., Mercer, C. D., & Miller, M. D. (2003). Teaching algebra to students with learning difficulties: An investigation of an explicit instruction model. *Learning Disabilities Research Practice, 18*(2), 121–131. https://doi.org/10.1111/1540-5826.00068
Xin, Y. P., Tzur, R., & Thouless, H. (Eds.). (2022). *Enabling mathematics learning of struggling students*. Springer.
Zbiek, R. M., & Conner, A. (2006). Beyond motivation: Exploring mathematical modeling as a context for deepening students' understandings of curricular mathematics. *Educational Studies in Mathematics, 63*, 89–112. https://doi.org/10.1007/s10649-005-9002-4
Zevenbergen, R. (2003). Ability grouping in mathematics classrooms: A Bourdieuian analysis. *For the Learning of Mathematics, 23*(3), 5–10. ⟨https://www.jstor.org/stable/40248425⟩.